of the order $t_{**} - t_* \gg \tau_0$ the process s(t) can be treated as rapidly varying in composition with $s_{**}(t|t_*)$, and within small error limits the stochastic interdependence of the values of these processes at coinciding times can be neglected. These assumptions are better satisfied the stronger the inequality (2.7). Under the stated assumptions the characteristic $\lambda(t|t_*)$ obeys the approximate expression

$$\lambda(t|t_*) \approx \int_0^\infty \int_0^\infty p_1(s_{**}, \dot{s}; t)\, p_2(s_{**}; t|t_*)\, \dot{s} d\dot{s} ds_{**}.$$

Here $v_1(s, \dot{s}; t)$ is the joint density function for the process s(t) and its first derivative $\dot{s}(t)$ at coinciding times, and $p_2(s_{**}; t|t_*)$ is the density function for the process $s_{**}(t|t_*)$ at the same times. The density function $p_1(s, \dot{s}; t)$ is specified as part of the description of the loading process, whereas the distribution of the values of the process $s_{**}(t|t_*)$, according to (4.2), is expressed in terms of the distribution function of the values of the process $\zeta(t|t_*)$.

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CONTINUATION WITH RESPECT TO A PARAMETER IN NONLINEAR ELASTICITY THEORY PROBLEMS

É. I. Grigolyuk and V. I. Shalashilin

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1. The equations describing the nonlinear static deformation of elastic systems generally contain a parameter, usually the load. We consider algebraic and transcendental equations. The generalization to functional and operator equations presents no difficulties in principle.

Let us consider a system of nonlinear equations for the vector $\mathbf{x} = \{x_1, \ldots, x_m\}$ containing a parameter λ:

$$F(\mathbf{x}, \lambda) = 0, \tag{1.1}$$

where $F = \{F_1(\mathbf{x}, \lambda), \ldots, F_m(\mathbf{x}, \lambda)\}$ is a vector function which is nonlinear with respect to $\mathbf{x}$ and λ, and is assumed continuous and differentiable with respect to $\mathbf{x}$ and λ a sufficient number of times.

Suppose for $\lambda \in [\lambda_0, \lambda_n]$ Eq. (1.1) has the solution $\mathbf{x}(\lambda)$, and that for $\lambda = \lambda_0$ the solution $\mathbf{x}_0 = \mathbf{x}(\lambda_0)$ is known, i.e.,

$$F(\mathbf{x}_{(0)}, \lambda_0) = 0. \tag{1.2}$$

We introduce an (m + 1)-dimensional vector space E_{m+1}: $\{\mathbf{x}, \lambda\}$. In this respect the point corresponding to the solution of (1.1) describes a continuous curve K which passes through the points $\mathbf{x}_{(0)}$, $\lambda_{(0)}$, and $\mathbf{x}_{(n)}$, $\lambda_{(n)}$. The idea of the method of continuation with respect to a parameter consists in constructing a sequence of solutions $\mathbf{x}_{(k)} = \mathbf{x}(\lambda_k)$ (k =

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1, ..., n) based on the known solution $\mathbf{x}_{(0)}$ for motion along curve K. Two forms of the method of continuation with respect to a parameter are known: continuous continuation [1] and discrete continuation [2].

2. The process of continuous continuation is based on the fact that the differentiation of Eqs. (1.1) with respect to the parameter λ leads to a system of linear equations for $dx_i/d\lambda$:

$$J\frac{d\mathbf{x}}{d\lambda}+\frac{\partial F}{\partial\lambda}=0,\quad J=\left\{F_{i,j}=\frac{\partial F_i}{\partial x_j}\right\}\quad (i,j=1,\ldots,m). \tag{2.1}$$

If for $\lambda\in[\lambda_0,\lambda_n]$ the Jacobian det (J) $\neq$ 0, the process of constructing the solution of system (1.1) is reduced to the integration of the Cauchy problem with respect to the parameter λ:

$$\frac{d\mathbf{x}}{d\lambda}=-J^{-1}\frac{\partial F}{\partial\lambda},\quad \mathbf{x}(\lambda_0)=\mathbf{x}_{(0)}. \tag{2.2}$$

Known explicit and implicit schemes can be used for the integration. However, all these schemes become useless near points where det (J) = 0, i.e., near the singular points of the curve K. We consider only the limit points. Ordinarily these difficulties are avoided by changing the parameter [1], using instead of (2.1) an analogous system which is obtained from (1.1) by differentiating with respect to one of the components x_t of the vector $\mathbf{x}$, and setting $\lambda=\lambda(x_t)$. This raises the question of choosing the optimum continuation parameter [3, 4].

We present a different approach. In E_{m+1}, we denote x_{m+1} by λ. Then Eqs. (1.1) take the form

$$F_i(\mathbf{x})=0\ (i=1,\ldots,m;\ \mathbf{x}\in E_{m+1}). \tag{2.3}$$

We write Eqs. (2.1) in the form of a system of m linear homogeneous equations for the m + 1 unknowns dx_j:

$$\sum_{j=1}^{m+1}F_{i,j}\,dx_j=0\quad (i=1,\ldots,m). \tag{2.4}$$

The matrix of this system $J_+=\{J, F, \lambda\}$, which is formed by adding the column vector F_λ to the right of J, has the important property that at both regular and limit points the rank of (J_+) is m. The vector $d\mathbf{x}=\{dx_1,\ldots,dx_m, dx_{m+1}=d\lambda\}$ by its very meaning is tangent to the curve K in E_{m+1}, and belongs to the one-dimensional subspace of the solutions of system (2.4). Henceforth, by $d\mathbf{x}$ we shall understand a unit vector of this subspace. Being a solution of system (2.4), it is orthogonal to the rows of the matrix J_+, and can be obtained as an orthogonal complement of the orthonormal basis of the rows of J_+ constructed by the Gramm–Schmidt process. Since the rank of (J_+) is m, such a basis exists at both regular and limit points. This method of solving systems (2.1), (2.4) is commonly called the orthogonalization method [5]. In the number of necessary operations and stability it is only slightly inferior to Gauss' method. However, when it is applied to continuation with respect to a parameter there is a considerable advantage of system (2.4) over (2.2) at each step, since continuation along the direction of $d\mathbf{x}$ is optimum [4]. The question of the optimum continuation parameter is solved in a natural way.

System (2.4) together with the initial condition $\mathbf{x}(\lambda_0)=\mathbf{x}_{(0)}$ is an implicit formulation of the Cauchy problem for $d\mathbf{x}$, and can be integrated by the same explicit and implicit schemes as problem (2.2).

3. The method of discrete continuation with respect to a parameter in the form proposed in [2] consists essentially in finding the solutions $\mathbf{x}_{(k)}=\mathbf{x}(\lambda_k)$ of system (1.1) by Newton's method, taking the solution $\mathbf{x}_{(k-1)}=\mathbf{x}(\lambda_{k-1})$ as the initial approximation. The corresponding iterative process has the form

$$\begin{gathered}\mathbf{x}_{(k)}^{(0)}=\mathbf{x}_{(k-1)},\quad J\left(\mathbf{x}_{(k)}^{(i)},\lambda_k\right)\cdot\Delta\mathbf{x}_{(k)}^{(i+1)}+F\left(\mathbf{x}_{(k)}^{(i)},\lambda_k\right)=0,\\ \mathbf{x}_{(k)}^{(i+1)}=\mathbf{x}_{(k)}^{(i)}+\Delta\mathbf{x}_{(k)}^{(i+1)},\quad i=0,1,\ldots\end{gathered} \tag{3.1}$$

Figure 1 illustrates this process geometrically for an equation with one unknown $F(\mathbf{x},\lambda)=0$. The solutions $\mathbf{x}_{(k)}$ sought lie on curve K along which the surface $F(\mathbf{x},\lambda)$ intersects the $\mathbf{x}$, λ

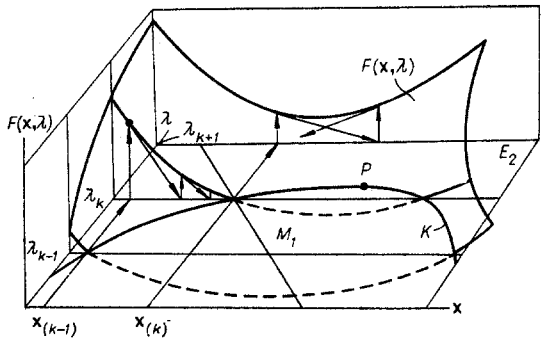

Fig. 1

plane. In the neighborhood of limit point P the transition from λ_k to λ_{k+1} removes process (3.1) from the domain where a solution exists, and it ceases to converge. In other words, the difficulty arises as a result of seeking the solution of (1.1) in the m-dimensional plane $\lambda = \lambda_{k+1} \in E_{m+1}$, which does not intersect K. It would be optimum to seek the solution $\mathbf{x}_{(k)}$ in the m-dimensional surface $M_m \in E_{m+1}$ which is orthogonal to K for $\mathbf{x} = \mathbf{x}_{(k)}$, but M_m is unknown until $\mathbf{x}_{(k)}$ is found. However, it is possible to seek the solution in a surface M^*_m near M_m. Let us consider some methods for prescribing M^*_m. To do this we use the equations in the form (2.3), regarding the parameter λ on the same basis as the remaining unknowns.

Let t be the size of the step with which we attempt to move along K. Then the plane M^*_m passing through the point $(\mathbf{x}_{(k-1)} + t d\mathbf{x}_{(k-1)}) \in E_{m+1}$ so that it is orthogonal to $d\mathbf{x}_{(k-1)}$ will be near M_m for small t. It is determined by the vector equation $d\mathbf{x}_{(k-1)} \cdot (\mathbf{x} - \mathbf{x}_{(k-1)} - t d\mathbf{x}_{(k-1)}) = 0$. Here the dot $\cdot$ denotes scalar multiplication. In this way the determination of the solution $\mathbf{x}_{(k)}$ following $\mathbf{x}_{(k-1)}$ is reduced to finding the solution of (2.3) in M^*_m, i.e., to the simultaneous solution of the equations

$$F_q(\mathbf{x}) = 0 \ (q = 1, \ldots, m; \ \mathbf{x} \in E_{m+1}); \tag{3.2}$$

$$d\mathbf{x}_{(k-1)} \cdot \mathbf{x} - (d\mathbf{x}_{(k-1)} \cdot \mathbf{x}_{(k-1)} + t) = 0. \tag{3.3}$$

Newton's iterative process for solving system (3.2) with condition (3.3) is illustrated in Fig. 2, and has the form

$$\mathbf{x}^{(0)}_{(k)} = \mathbf{x}_{(k-1)} + t d\mathbf{x}_{(k-1)}; \tag{3.4}$$

$$\begin{cases} J_+(\mathbf{x}^{(i)}_{(k)}) \cdot \Delta\mathbf{x}^{(i+1)}_{(k)} + F(\mathbf{x}^{(i)}_{(k)}) = 0, \\ d\mathbf{x}_{(k-1)} \cdot \Delta\mathbf{x}^{(i+1)}_{(k)} + d\mathbf{x}_{(k-1)} \cdot (\mathbf{x}^{(i)}_{(k)} - \mathbf{x}_{(k-1)}) - t = 0; \end{cases} \tag{3.5}$$

$$\mathbf{x}^{(i+1)}_{(k)} = \mathbf{x}^{(i)}_{(k)} + \Delta\mathbf{x}^{(i+1)}_{(k)}, \quad i = 0, 1, \ldots \tag{3.6}$$

We can understand the second of Eqs. (3.5) as the requirement for the correction vectors $\Delta\mathbf{x}^{(i+1)}_{(k)}$ to be orthogonal to $d\mathbf{x}_{(k-1)}$, and write it in the form $d\mathbf{x}_{(k-1)} \cdot \Delta\mathbf{x}^{(i+1)}_{(k)} = 0$.

If we introduce the vector $d\mathbf{x}^{(i)}_{(k)} = (\mathbf{x}^{(i)}_{(k)} - \mathbf{x}_{(k-1)})/t$, then by analogy the second of Eqs. (3.5) can be replaced by

$$d\mathbf{x}^{(i)}_{(k)} \cdot \Delta\mathbf{x}^{(i+1)}_{(k)} = 0. \tag{3.7}$$

The geometry of the process with such a condition is shown in Fig. 3 by the dashed lines. M^*_m is corrected at each step. The process shown in Fig. 3 by solid lines seems still more effective. It is realized by the following algorithm:

$$d\mathbf{x}^{(0)}_{(k)} = d\mathbf{x}_{(k-1)}, \quad \mathbf{x}^{(0)}_{(k)} = \mathbf{x}_{(k-1)} + t d\mathbf{x}^{(0)}_{(k)}; \tag{3.8}$$

$$\begin{cases} J_+(\mathbf{x}^{(i)}_{(k)}) \cdot \Delta\mathbf{x}^{(i+1)}_{(k)} + F(\mathbf{x}^{(i)}_{(k)}) = 0, \\ d\mathbf{x}^{(i)}_{(k)} \cdot \Delta\mathbf{x}^{(i+1)}_{(k)} = 0; \end{cases} \tag{3.9}$$

$$d^{\xi}_{S(k)}{}^{(i+1)} = d\mathbf{x}^{(i)}_{(k)} + \Delta\mathbf{x}^{(i+1)}_{(k)}, \quad d\mathbf{x}^{(i+1)}_{(k)} = d^{\xi}_{S(k)}{}^{(i+1)} / \left(d^{\xi}_{S(k)}{}^{(i+1)} \cdot d^{\xi}_{S(k)}{}^{(i+1)}\right)^{\frac{1}{2}}; \tag{3.10}$$

$$\mathbf{x}^{(i+1)}_{(k)} = \mathbf{x}_{(k-1)} + t d\mathbf{x}^{(i+1)}_{(k)}, \quad i = 0, 1, \ldots \tag{3.11}$$

This algorithm ensures the size of the step t along a chord of curve K with the specified accuracy.

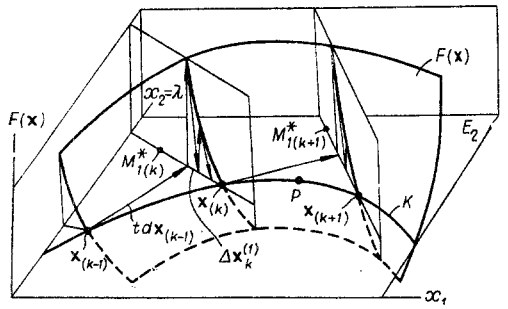

Fig. 2

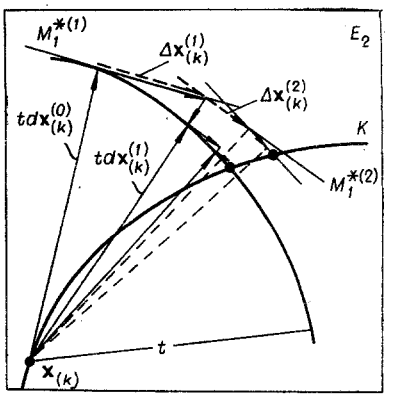

Fig. 3

All the algorithms presented above permit replacing the vector $d\mathbf{x}_{(k-1)}$ by its approximate value $d\mathbf{x}^*_{(k-1)} = (\mathbf{x}_{(k-1)} - \mathbf{x}_{(k-2)})/[(\mathbf{x}_{(k-1)} - \mathbf{x}_{(k-2)})\,(\mathbf{x}_{(k-1)} - \mathbf{x}_{(k-2)})]^{1/2}$, and the usual modification of Newton's method with the replacement of $J_+(\mathbf{x}^{(i)}_{(k)})$ by $J_+(\mathbf{x}^{(0)}_{(k)})$.

We note that an approach similar to the geometric treatment with process (3.4)-(3.6) was discussed in [4], but the supplementary equation proposed there is cumbersome and requires an appreciable number of additional calculations, as noted by the authors.

The surface M^*_m may not be a plane. For example, if we seek a solution for (2.3) on a sphere of radius t with its center at $\mathbf{x}_{(k+1)}$, we have to solve simultaneously the equations

$$F_q(\mathbf{x}) = 0 \ (q = 1, \ldots, m), \ (\mathbf{x} - \mathbf{x}_{(k-1)})\cdot(\mathbf{x} - \mathbf{x}_{(k-1)}) - t^2 = 0.$$

The algorithm of Newton's method takes the form (3.4)-(3.6) with the replacement of the supplementary equation in (3.5) by

$$d\mathbf{x}^{(i)}_{(k)}\cdot\Delta\mathbf{x}^{(i+1)}_{(k)} + \frac{t}{2}\left(d\mathbf{x}^{(i)}_{(k)}\cdot d\mathbf{x}^{(i)}_{(k)} - 1\right) = 0.$$

The solution of Eqs. (3.5) and (3.9) by the orthogonalization method eliminates computational difficulties at both regular and limit points.

In conclusion, we note that the continuous and discrete continuation processes formulated here have a considerable algorithmic generality, which makes it convenient to combine continuous and discrete continuation, using the latter of necessity for periodic refinement of the solution.

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